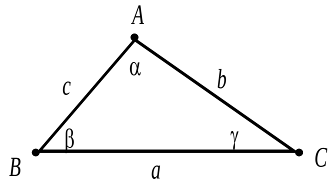
The Law of Cosines



The Law of Cosines: In any triangle,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Theorem 7.4. The Law of Cosines. Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following equations hold

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha) \qquad b^2 = a^2 + c^2 - 2ac\cos(\beta) \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

SSS: Use the fact that the largest angle is across from the longest side of the triangle to solve for the largest angle using the law of cosines. (For example, if c is the longest side, use the equation $c^2 = a^2 + b^2 - 2ab\cos\gamma$ to solve for γ.) Then use the law of sines to find the remaining angles, which will both be acute.

Example: $a = 3.8,\ b = 9.6,\ c = 7.7$

SAS: Find the length of the third side using the law of cosines. Use the law of sines to find the angle across from the shorter of the two given sides. Find the remaining angle by subtracting the first two from 180°.

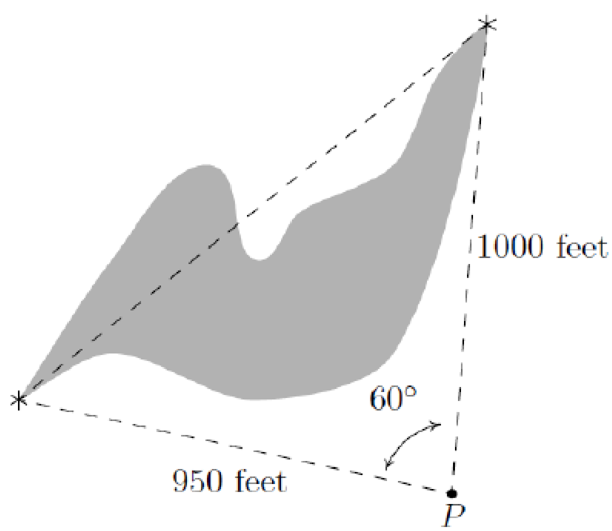
Example: $b = 5.8,\ c = 3.6,\ \alpha = 39.5^\circ$

Example: Jan and Dean started hiking from the same location at the same time. Jan hiked at 4 mph with bearing N12°E, and Dean hiked at 5 mph with bearing N31°W. How far apart were they after 6 hours? Round to the nearest tenth of a mile.

Solving Applied Problems Using the Law of Cosines

Next, we have an application of the Law of Cosines.

Example 7.3.3. A researcher wishes to determine the width of a vernal pond as drawn below. From a point P, he finds the distance to the western-most point of the pond to be 950 feet, while the distance to the northern-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is $60°$, find the width of the pond.



Solution. We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle. Calling this length w (for width), we get

$$\begin{aligned} w^2 &= 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) \\ &= 952{,}500 \end{aligned}$$

We next take the square root to get

$$\begin{aligned} w &= \sqrt{952500} \\ &\approx 976 \text{ feet} \end{aligned}$$

□

Herons Formula:

Using the law of cosines, it is possible to derive a formula for the area of a triangle that involves only the lengths of the sides of the triangle. The formula is known as "Heron's Formula" after Heron of Alexandria, who is believed to have discovered it around AD 75.

Heron's Formula: The area of a triangle with sides of lengths *a*, *b*, and *c* is given by:

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = (a+b+c)/2.$$

Find the area of the triangle with $a = 12,\ b = 8,$ and $c = 6$

Find the area of a triangle with $a = 346,\ b = 234,$ and $c = 422$